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## Tau Decays into Kaons\*

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### Abstract

Predictions for semi-leptonic decay rates of the  $\tau$  lepton into two meson final states  $K^-\pi^0\nu_\tau$ ,  $\overline{K}^0\pi^-\nu_\tau$ ,  $K^0K^-\nu_\tau$ , and three meson final states  $K^-\pi^-K^+\nu_\tau$ ,  $K^0\pi^-\overline{K}^0\nu_\tau$ ,  $K_S\pi^-K_S\nu_\tau$ ,  $K_S\pi^-K_L\nu_\tau$ ,  $K_L\pi^-K_L\nu_\tau$ ,  $K^-\pi^0K^0\nu_\tau$ ,  $\pi^0\pi^0K^-\nu_\tau$ ,  $K^-\pi^-\pi^+\nu_\tau$ ,  $\pi^-\overline{K}^0\pi^0\nu_\tau$  are derived. The hadronic matrix elements are expressed in terms of form factors, which can be predicted by chiral Lagrangians supplemented by informations about all possible low-lying resonances in the different channels. Isospin symmetry relations among the different final states are carefully taken into account. The calculated branching ratios are compared with measured decay rates where data are available.

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\*We wish to dedicate this work to Roger Decker whose death at an early age is a great loss to his friends and to the physics community.

## I. INTRODUCTION

With the experimental progress in  $\tau$  decays an ideal tool for studying strong interaction physics has been developed. In particular, final states with kaons provide a powerful probe of the strange sector of the weak charged current.

The decay rate for the simplest decay mode with one kaon  $\tau \rightarrow K\nu_\tau$  is well predicted by the kaon decay constant  $f_K$ , which can be measured in  $\Gamma(K \rightarrow \mu\bar{\nu}_\mu)$ . Radiative corrections to this decay mode are also available [1]. The decay rate for  $\tau \rightarrow K\pi\nu_\tau$  on the other hand is the Cabibbo suppressed analogy to  $\tau \rightarrow \rho(\rightarrow \pi\pi)\nu_\tau$ , *i.e.* the  $K\pi$  system is expected to be dominated by the  $K^*(892)$  resonance [2]. However, the results for this decay rate are fairly sensitive towards the details of the parameterization of the  $K^*$  resonance, such as contributions from higher radial excitations, and we will discuss this dependence in some detail in Sec. II. Since a reliable parameterization of the  $K^*$  propagator is also needed for the three meson decay modes discussed below, the informations from the  $\tau \rightarrow K\pi\nu_\tau$  decay rates are used to put constraints on the  $K^*$  resonance parameterization.

The hadronic matrix elements for three meson final states with pions and kaons have a much richer structure. A general parameterization of the form factors in these decay modes was proposed in [3]. The physical idea behind this model can be summarized to:

- In the chiral limit the form factors are normalized to the  $SU(3)_L \times SU(3)_R$  chiral model.
- Meson-vertices are independent of momentum.
- The full momentum dependence is given by normalized Breit-Wigner propagators of the resonances occurring in the different channels. Resonances occur either in  $Q^2$  [the total invariant hadronic mass squared] which are the three body resonances or in the Dalitz plot variables  $s_i$  which are two body resonances.

Parameterizations of the amplitude for the  $3\pi$  final states within this model can be found in [4–6]. In this case the vector form factor is absent due to the  $G$  parity of the pions. The pion decay modes offer a unique tool for the study of  $\rho, \rho'$  resonance parameters in different hadronic environments, competing well with low energy  $e^+e^-$  colliders with energies in the region below 1.7 GeV. The decay modes involving pions and kaons allow for axial and vector current contributions at the same time [7,3]. The vector form factor is related to the Wess-Zumino anomaly [8,9] whereas the axial-vector form factors are predicted by chiral Lagrangians as mentioned before. A parameterization of the  $\tau^- \rightarrow K^-\pi^-K^+\nu_\tau$  decay mode within this framework has been developed in [10]. However, our result for this decay mode differs from the their result (see Sec. VII).

In the present paper, we reexamine the model used in [3] for the  $\tau$  decay modes involving pions and kaons. We take into account possible isospin and  $SU(3)$  symmetry relations, which allow for additional resonance contributions to the form factors. We reanalyze the issue of the strange axial resonances ( $K_1$  states) in view of new experimental results, and we include  $\omega - \Phi$  mixing. The new parameterization leads to sizable differences in the predictions of the decay rates compared to [3]. In addition, we derive a parameterization for the final states with two neutral kaons  $\tau^- \rightarrow K_S\pi^-K_S\nu_\tau$ ,  $\tau^- \rightarrow K_L\pi^-K_L\nu_\tau$ , and  $\tau^- \rightarrow K_S\pi^-K_L\nu_\tau$ .

The paper is organized as follows: In Sec. II we discuss the two meson decay modes  $K^-\pi^0\nu_\tau$ ,  $\overline{K}^0\pi^-\nu_\tau$  and  $K^0K^-\nu_\tau$  and fix the parameters of the  $K^*$  resonance. In Sec. III we review the general structure of the matrix elements of the weak hadronic current in the three meson case. In Sec. IV (Sec. V) we discuss in detail the individual matrix elements of the axial-vector (vector) current, carefully exploiting isospin and flavour symmetry. Final states with two neutral kaons are considered in Sec. VI. We give our numerical results in Sec. VII, and finally in Sec. VIII we give a brief summary.

## II. THE TWO MESON DECAY MODES

As mentioned before, a reliable parameterization of the  $K^*$  propagator is needed for the three meson decay modes with kaons. Since the decay modes  $\tau \rightarrow K^-\pi^0\nu_\tau$  and  $\tau \rightarrow \overline{K}^0\pi^-\nu_\tau$  are dominated by the  $K^*$  resonance, we use the experimental informations on the decay rates into these final states to fix the parameters of the  $K^*$  resonance.

The matrix element for the semi-leptonic decay into two mesons  $h_a$  and  $h_b$

$$\tau(l, s) \rightarrow \nu(l', s') + h_a(q_1, m_1) + h_b(q_2, m_2) , \quad (1)$$

can be expressed in terms of a leptonic ( $M_\mu$ ) and a hadronic vector current ( $J^\mu$ ) as

$$\mathcal{M} = \frac{G}{\sqrt{2}} \begin{pmatrix} \cos \theta_c \\ \sin \theta_c \end{pmatrix} M_\mu J^\mu . \quad (2)$$

In Eq. (2),  $G$  denotes the Fermi-coupling constant and  $\theta_c$  is the Cabibbo angle. The leptonic and hadronic currents are given by

$$M_\mu = \bar{u}(l', s') \gamma_\mu (1 - \gamma_5) u(l, s) , \quad (3)$$

and

$$J^\mu(q_1, q_2) = \langle h_a(q_1) h_b(q_2) | V^\mu(0) | 0 \rangle . \quad (4)$$

The hadronic matrix elements for the Cabibbo suppressed decay modes  $K^-\pi^0\nu_\tau$ ,  $\overline{K}^0\pi^-\nu_\tau$  are dominated by the  $K^*$  resonance  $T_{K^*}^{(1)}(Q^2)$ , whereas the one for the Cabibbo allowed mode  $K^0K^-$  is dominated by the high energy tail of the  $\rho$ . One has

$$\begin{aligned} \langle K^-(q_1) \pi^0(q_2) | V^\mu | 0 \rangle &= \frac{1}{\sqrt{2}} \langle \overline{K}^0(q_1) \pi^-(q_2) | V^\mu | 0 \rangle , \\ &= \frac{1}{\sqrt{2}} T_{K^*}^{(1)}(Q^2) (q_1 - q_2)_\nu T^{\mu\nu} \\ \langle K^0(q_1) K^-(q_2) | V^\mu | 0 \rangle &= T_\rho^{(1)}(q_1 - q_2)_\nu T^{\mu\nu} , \end{aligned} \quad (5)$$

where  $Q = q_1 + q_2$  and  $T^{\mu\nu}$  denotes the transverse projector, defined by

$$T^{\mu\nu} = g^{\mu\nu} - \frac{Q^\mu Q^\nu}{Q^2} . \quad (6)$$

The form and the normalization of the hadronic matrix elements in Eq. (5) are fixed by chiral symmetry constraints, which determines the matrix elements in the limit of soft meson momenta  $Q^2 \rightarrow 0$ . The strong interaction effects beyond the low energy limit are taken into account by the vector resonance factors  $T_\rho^{(1)}(Q^2)$  and  $T_{K^*}^{(1)}(Q^2)$  with the requirement  $T_X^{(1)}(Q^2 = 0) = 1$  ( $X = \rho, K^*$ ). Note that in the case of the  $K^*$  we have neglected a scalar contribution proportional to  $Q^\mu$ . This scalar part is proportional to the off-shellness ( $m_{K^*}^2 - Q^2$ ) of the  $K^*$  and therefore strongly suppressed. We have checked its size numerically and found it to be negligible.

Note that our results for the hadronic matrix elements for  $K^-\pi^0$  and  $\bar{K}^0\pi^-$  in Eq. (5) differ from the results in the Tauola Monte Carlo [11] by an overall factor. In fact we believe that Tauola is off from the correct normalization by a factor of  $2/\sqrt{3}$ , *i.e.* the corresponding matrix elements in Tauola should be multiplied by  $\sqrt{3}/2$ . Note furthermore that Tauola does not include the  $K^*(1410)$  in the  $K\pi$  resonance, *i.e.* it corresponds to  $\beta_{K^*} = 0$  in Eq. (10). These two differences with our parameterization almost cancel each other, such that Tauola (with  $\beta_{K^*} = 0$ ) gives a number for the  $\tau \rightarrow K^*\nu_\tau$  branching ratio which is very close to our result with  $\beta_{K^*} = -0.135$  [see below].

If not stated otherwise, we use the following form for the two particle Breit-Wigner propagators with an energy dependent width  $\Gamma_X(s)$  throughout this paper:

$$\text{BW}_X[s] \equiv \frac{M_X^2}{[M_X^2 - s - i\sqrt{s}\Gamma_X(s)]}, \quad (7)$$

where  $X$  stands for the various resonances of the two meson channels. For a  $1 \rightarrow 2$  decay, the energy dependent width is [3]

$$\begin{aligned} \Gamma_X(s) &= \Gamma_X \frac{M_X^2}{s} \left( \frac{p}{p_X} \right)^{2n+1}, \\ p &= \frac{1}{2\sqrt{s}} \sqrt{(s - (M_1 + M_2)^2)(s - (M_1 - M_2)^2)}, \\ p_X &= \frac{1}{2M_X} \sqrt{(M_X^2 - (M_1 + M_2)^2)(M_X^2 - (M_1 - M_2)^2)}, \end{aligned}$$

where  $n$  is the power of  $|p|$  in the matrix element, *i.e.*  $n = 1$  for the decay modes in this paper.

We use the following parameterizations for the  $\rho$  resonance:

$$T_\rho^{(1)}(s) = \frac{1}{1 + \beta_\rho} [\text{BW}_\rho(s) + \beta_\rho \text{BW}_{\rho'}(s)], \quad (8)$$

where

$$\begin{aligned} \beta_\rho &= -0.145, \\ m_\rho &= 0.773 \text{ GeV}, \Gamma_\rho = 0.145 \text{ GeV}, \\ m_{\rho'} &= 1.370 \text{ GeV}, \Gamma_{\rho'} = 0.510 \text{ GeV}. \end{aligned} \quad (9)$$

These are the values which have been determined from  $e^+e^- \rightarrow \pi^+\pi^-$  in [4] and have been used in [3] for the non-strange case. For the vector resonances with strangeness, only the

$K^*(892)$  was considered in [3]. Our parameterization for  $T_{K^*}^{(1)}(Q^2)$  allows for a contribution of the first excitation  $K^{*'}(1410)$  in analogy to Eq. (8):

$$T_{K^*}^{(1)}(s) = \frac{1}{1 + \beta_{K^*}} [\text{BW}_{K^*}(s) + \beta_{K^*} \text{BW}_{K^{*'}}(s)] , \quad (10)$$

where

$$\begin{aligned} m_{K^*} &= 0.892 \text{ GeV} , \Gamma_{K^*} = 0.050 \text{ GeV} , \\ m_{K^{*'}} &= 1.412 \text{ GeV} , \Gamma_{K^{*'}} = 0.227 \text{ GeV} . \end{aligned} \quad (11)$$

In the limit of  $SU(3)$  flavour symmetry, one would expect a contribution to the  $K^*$  resonance from the first excitation  $K^{*'}(1410)$  with the same relative strength  $\beta_{K^*} = -0.145$  as measured in the non-strange case. The numerical results for the decay rates  $K^-\pi^0\nu_\tau$ ,  $\overline{K}^0\pi^-\nu_\tau$  are very sensitive to the parameter  $\beta_{K^*}$  in Eq. (10). Since a reliable parameterization of the  $K^*$  propagator will be needed for the decay modes considered in this paper, we will use the decay mode  $\tau \rightarrow K^*\nu_\tau$  to fix the parameter  $\beta_{K^*}$ .

Adding up the two charge modes in Eq. (5)

$$\mathcal{B}(K^*\nu_\tau) = \mathcal{B}(K^-\pi^0\nu_\tau) + \mathcal{B}(\overline{K}^0\pi^-\nu_\tau) , \quad (12)$$

we obtain

$$\begin{aligned} \mathcal{B}(K^*\nu_\tau) &= 1.00\% & \text{for} & \quad \beta_{K^*} = 0 , \\ \mathcal{B}(K^*\nu_\tau) &= 1.28\% & \text{for} & \quad \beta_{K^*} = -0.11 , \\ \mathcal{B}(K^*\nu_\tau) &= 1.36\% & \text{for} & \quad \beta_{K^*} = -0.135 , \\ \mathcal{B}(K^*\nu_\tau) &= 1.44\% & \text{for} & \quad \beta_{K^*} = -0.157 . \end{aligned} \quad (13)$$

The main effect is due to the normalization factor  $1/(1 + \beta_{K^*})^2$  multiplying the  $K^*(892)$  contribution, whereas the  $K^*(1410)$  contribution is strongly phase space suppressed.

The latest experimental result on the branching fraction  $\mathcal{B}(K^*\nu_\tau)$  is  $1.36 \pm 0.08 \%$  [12]. Thus our results favour a negative value of  $\beta_{K^*}$ , and we obtain  $\beta_{K^*} = -0.135 \pm 0.025$  as a result of this analysis. This value is remarkably close to the strength of the  $\rho'$  contribution to the  $\rho$  Breit-Wigner in Eqs. (8,9), supporting the use of approximate  $SU(3)$  flavour symmetry.

Note that we use here the values of [13] for the mass and widths parameters of the  $K^{*'}$  in Eqs. (10,11), whereas the  $\rho'$  mass and width parameters in Eq. (9) have been determined from a fit to the  $Q^2$  distribution in  $\tau \rightarrow 2\pi\nu_\tau$  [4]. Indeed, a more reliable determination of the parameters of the off-shell  $K^*$  propagator (including  $\beta_{K^*}$ ) could be obtained from a fit to the  $Q^2$  distribution in  $\tau \rightarrow K^*\nu_\tau$ . Unfortunately, there is no experimental information of the  $Q^2$  dependence in this decay mode available, and we use therefore the simpler approach of fixing  $\beta_{K^*}$  from the branching ratio and using the  $K^{*'}$  parameters from [13].

For the decay into two kaons we obtain from the matrix element in Eq. (5)

$$\mathcal{B}(K^0K^-\nu_\tau) = 0.11\% , \quad (14)$$

in good agreement with the recent world average  $\mathcal{B}(K^0K^-\nu_\tau) = 0.13 \pm 0.04\%$  [12].

### III. GENERAL STRUCTURE OF THE WEAK MATRIX ELEMENTS IN THE THREE MESON MODES

Let us briefly recapitulate the general structure of the semi-leptonic decay

$$\tau(l, s) \rightarrow \nu(l', s') + h_a(q_1, m_1) + h_b(q_2, m_2) + h_c(q_3, m_3), \quad (15)$$

as introduced in [5]. In Eq. (15),  $h(q_i, m_i)$  are pseudoscalar mesons and in our case of interest, at least one of the mesons is a kaon. The matrix element is

$$\mathcal{M} = \frac{G}{\sqrt{2}} \begin{pmatrix} \cos \theta_c \\ \sin \theta_c \end{pmatrix} M_\mu J^\mu, \quad (16)$$

where the cosine and the sine of the Cabibbo angle ( $\theta_C$ ) in Eq. (16) have to be used for Cabibbo allowed  $\Delta S = 0$  [*i.e.* final states with two kaons] and Cabibbo suppressed  $|\Delta S| = 1$  [*i.e.* final states with one kaon] decays, respectively. The leptonic ( $M_\mu$ ) current is already given in Eq. (3) and the hadronic ( $J^\mu$ ) currents can be written as

$$J^\mu(q_1, q_2, q_3) = \langle h_a(q_1) h_b(q_2) h_c(q_3) | V^\mu(0) - A^\mu(0) | 0 \rangle. \quad (17)$$

$V^\mu$  and  $A^\mu$  are the vector and axial-vector quark currents, respectively. The most general ansatz for the matrix element of the quark current  $J^\mu$  in Eq. (17) is characterized by four form factors

$$J^\mu(q_1, q_2, q_3) = V_1^\mu F_1 + V_2^\mu F_2 + i V_3^\mu F_3 + V_4^\mu F_4, \quad (18)$$

with

$$\begin{aligned} V_1^\mu &= (q_1 - q_3)_\nu T^{\mu\nu}, \\ V_2^\mu &= (q_2 - q_3)_\nu T^{\mu\nu}, \\ V_3^\mu &= \epsilon^{\mu\alpha\beta\gamma} q_{1\alpha} q_{2\beta} q_{3\gamma}, \\ V_4^\mu &= q_1^\mu + q_2^\mu + q_3^\mu = Q^\mu. \end{aligned} \quad (19)$$

$T^{\mu\nu}$  denotes the transverse projector, defined in Eq. (6). Since the strong interaction conserves parity, the axial-vector current induces the form factors  $F_1, F_2$  and  $F_4$  while the vector current induces  $F_3$ . In the limit of vanishing quark masses, the weak axial-vector current is conserved and this implies that the scalar form factor  $F_4$  vanishes. The massive pseudoscalars give a contribution to  $F_4$ , however, the effect is very small [14] and we will neglect this contribution in the subsequent discussion, *i.e.* we set  $F_4$  equal to zero. All form factors  $F_i$  are in general functions of  $Q^2$ ,  $s_1 = (q_2 + q_3)^2$ ,  $s_2 = (q_1 + q_3)^2$  and  $s_3 = (q_1 + q_2)^2$ .

A specific model for the form factors  $F_i$  for various three meson final states was derived in [3]. This model takes into account the chiral symmetry constraints of QCD as well as the resonance phenomena present in  $\tau$  decays and has been developed in [15]. Let us briefly summarize the results (for more details see [3]): The chiral symmetry constraints lead to the following expression for the hadronic axial-vector current in the limit of soft meson momenta:

$$\langle h_a h_b h_c | A^\mu | 0 \rangle = \frac{2\sqrt{2} A^{(abc)}}{3f_\pi} \left\{ G_{1,soft}^{(abc)} (q_1 - q_3)_\nu + G_{2,soft}^{(abc)} (q_2 - q_3)_\nu \right\} T^{\mu\nu}, \quad (20)$$

where the coefficients  $A^{(abc)}$  are given in Tab. I for the various decay modes and the coefficients  $G_{i,soft}^{(abc)}$  are all equal to one

$$G_{i,soft}^{(abc)} = 1, \quad (i = 1, 2) \quad (21)$$

except for  $G_{2,soft}^{(K^-\pi^0 K^0)}$  and  $G_{2,soft}^{(\pi^-\bar{K}^0 \pi^0)}$  which vanish:  $G_{2,soft}^{(K^-\pi^0 K^0)} = G_{2,soft}^{(\pi^-\bar{K}^0 \pi^0)} = 0$ . The vector current arises from the Wess-Zumino Lagrangian for the axial anomaly [8,9]. One obtains in the low energy limit:

$$\langle h_a h_b h_c | V^\mu | 0 \rangle = \frac{i}{2\sqrt{2}\pi^2 f_\pi^3} A^{(abc)} \epsilon^{\mu\alpha\beta\gamma} q_{1\alpha} q_{2\beta} q_{3\gamma} G_{3,soft}^{(abc)}. \quad (22)$$

The coefficients  $A^{(abc)}$  for the vector current are given in Tab. II and  $G_{3,soft}^{(abc)} = 1$ , except for  $G_{3,soft}^{(K^-\pi^0 K^0)}$  and  $G_{3,soft}^{(\pi^0 \pi^0 K^-)}$  which vanish:  $G_{3,soft}^{(K^-\pi^0 K^0)} = G_{3,soft}^{(\pi^0 \pi^0 K^-)} = 0$ .

As mentioned before, the strong interaction effects beyond the low energy limit are taken into account by inserting resonance form factors  $G_{1,2,3}(Q^2, s_i)$  [ $i = 1, 2, 3$ ] into the amplitudes of Eqs. (20,22) with the requirement  $G_{1,2,3}^{(abc)} \rightarrow G_{1,2,3,soft}^{(abc)} = 1$  in the limit  $Q^2, s_i \rightarrow 0$ , except for  $G_{2,3}^{(K^-\pi^0 K^0)}$ ,  $G_2^{(\pi^-\bar{K}^0 \pi^0)}$  and  $G_3^{(\pi^0 \pi^0 K^-)}$ , which vanish in the chiral limit.

In fact, the functions  $G_{1,2,3}^{(abc)}$  are products of normalized Breit-Wigner resonances in  $Q^2$  and  $s_i$ . Comparing Eqs. (20,22) with the general expansion in Eqs. (18,19) leads to

$$F_1^{(abc)}(Q^2, s_2, s_3) = \frac{2\sqrt{2}A^{(abc)}}{3f_\pi} G_1^{(abc)}(Q^2, s_2, s_3), \quad (23)$$

$$F_2^{(abc)}(Q^2, s_1, s_3) = \frac{2\sqrt{2}A^{(abc)}}{3f_\pi} G_2^{(abc)}(Q^2, s_1, s_3), \quad (24)$$

$$F_3^{(abc)}(Q^2, s_1, s_2, s_3) = \frac{A^{(abc)}}{2\sqrt{2}\pi^2 f_\pi^3} G_3^{(abc)}(Q^2, s_1, s_2, s_3). \quad (25)$$

The Breit-Wigner functions  $G_{1,2}$  ( $G_3$ ) are listed in Tab. I (II) for the various decay modes and the precise form will be discussed in the subsequent sections. Note that by convenient ordering of the mesons, the two body resonances in  $F_1$  ( $F_2$ ) occur only in the variables  $s_2, s_3$  ( $s_1, s_3$ ).

#### IV. ISOSPIN AND FLAVOUR SYMMETRY RELATIONS IN THE AXIAL-VECTOR CURRENT

Amongst the hadronic matrix elements for the hadronic final states  $K^-\pi^-K^+$ ,  $K^0\pi^-\bar{K}^0$  and  $K^-\pi^0 K^0$  (which are versions of  $K\pi\bar{K}$  with different third components of the isospin) and amongst those for  $\pi^0\pi^0 K^-$ ,  $K^-\pi^-\pi^+$  and  $\pi^-\bar{K}^0\pi^0$  (different versions of  $\bar{K}\pi\pi$ ), there are isospin symmetry relations which have not been fully exploited in [3]. Consider for example the final state  $K^-\pi^-K^+$ , which is dominated by a  $(K^*)^0$  resonance in  $\pi^-K^+$ . In Tab. I of [3], this is taken into account by

$$G_2^{(K^-\pi^-K^+)} = \text{BW}_{A_1}(Q^2) T_{K^*}(s_1). \quad (26)$$

However, by isospin symmetry, the amplitudes  $(K^*)^0 \rightarrow \pi^0 K^0$  and  $(K^*)^0 \rightarrow \pi^- K^+$  have a ratio of  $\sqrt{1/3} : (-)\sqrt{2/3}$ . Taking into account the different normalization coefficients  $A^{(abc)}$ , this immediately leads to

$$G_2^{(K^-\pi^0 K^0)} = \text{BW}_{A_1}(Q^2) \left( \frac{1}{3} T_{K^*}(s_1) + \dots \right), \quad (27)$$

which differs from  $G_2^{(K^-\pi^0 K^0)} = 0$  in Tab. I of [3]. The dots indicate that there must be an additional contribution which ensures that  $G_2^{(K^-\pi^0 K^0)} \rightarrow 0$  for  $Q^2, s_i \rightarrow 0$  as required by the chiral limit (see below).

In fact, by using all the relevant Clebsch-Gordon coefficients, we can predict the matrix element for  $K^-\pi^0 K^0$  from those for  $K^-\pi^- K^+$  and  $K^0\pi^-\bar{K}^0$ , and the one for  $\pi^-\bar{K}^0\pi^0$  can be predicted from those of  $\pi^0\pi^0 K^-$  and  $K^-\pi^-\pi^+$ . In using the Clebsch-Gordons, some care is needed because in [3], the phase conventions are *not* identical to the Condon-Shortley-de Swart ones. And so, in a first step Tab. I of [3] has to be transformed into Condon-Shortley-de Swart by multiplying  $A^{(abc)}$  with  $(-1)$  for each  $\pi^-, K^-, K^0, \bar{K}^0$  in the final state. Then the Clebsch-Gordans may be used, and finally the results are transformed back into the conventions of [3]. This issue does not affect the above example, because  $\langle K^-\pi^- K^+ |$  and  $\langle K^-\pi^0 K^0 |$  are even under this transformation.

Taking the matrix elements for  $K^-\pi^- K^+$  and  $K^0\pi^-\bar{K}^0$ ,  $\pi^0\pi^0 K^-$  and  $K^-\pi^-\pi^+$  from [3], we obtain (see also Tab. I)

$$\begin{aligned} \langle K^-\pi^0 K^0 | A^\mu | 0 \rangle &= \frac{2\sqrt{2}}{3f_\pi} \frac{3 \cos \theta_c}{2\sqrt{2}} \text{BW}_{A_1}(Q^2) T^{\mu\nu} \\ &\times \left\{ \left[ \frac{2}{3} T_\rho^{(1)}(s_2) + \frac{1}{3} T_{K^*}(s_3) \right] (q_1 - q_3)_\nu + \frac{1}{3} [T_{K^*}(s_1) - T_{K^*}(s_3)] (q_2 - q_3)_\nu \right\}, \end{aligned} \quad (28)$$

$$\begin{aligned} \langle \pi^-\bar{K}^0\pi^0 | A^\mu | 0 \rangle &= \frac{2\sqrt{2}}{3f_\pi} \frac{3 \sin \theta_c}{2\sqrt{2}} \text{BW}_{K_1}(Q^2) T^{\mu\nu} \\ &\times \left\{ \left[ \frac{2}{3} T_\rho^{(1)}(s_2) + \frac{1}{3} T_{K^*}(s_3) \right] (q_1 - q_3)_\nu + \frac{1}{3} [T_{K^*}(s_1) - T_{K^*}(s_3)] (q_2 - q_3)_\nu \right\}. \end{aligned} \quad (29)$$

Note that the form factor  $G_2$  in the two decay modes  $K^-\pi^0 K^0$  and  $\pi^-\bar{K}^0\pi^0$  receive contributions from  $T_{K^*}$  resonances, which however vanish in the low energy chiral limit, where our Breit-Wigner propagators are normalized to one. In [3], this contribution was not taken into account.

The parameterizations of the two particle resonances have been determined in Sec. II, where we found  $\beta_{K^*} = -0.135 \pm 0.025$  for the  $K^*$  resonance. Our numerical results for the three meson decay modes are very sensitive to the parameter  $\beta_{K^*}$  and we will discuss this dependence in more detail in Sec. VII.

Let us now discuss the three particle resonances. As in [3], we use the  $A_1$  resonance in the non-strange case with energy dependent width

$$\text{BW}_{A_1}(s) = \frac{m_{A_1}^2}{m_{A_1}^2 - s - im_{A_1} \Gamma_{A_1} g(s)/g(m_{A_1})}, \quad (30)$$



$$m_{A_1} = 1.251 \text{ GeV} , \quad \Gamma_{A_1} = 0.475 \text{ GeV} . \quad (31)$$

where the function  $g(s)$  has been calculated in [4]. Note that we use a smaller  $A_1$  width than used in [3,5,6] ( $\Gamma_{A_1} = 0.599 \text{ GeV}$ ). The formalism of structure functions, which was developed in [5,6], allow for a much more detailed test of the hadronic matrix elements and the resonance parameters than it is possible by a rate measurement alone. However, the ratios of structure functions, which were predicted in [5,6] for the  $\tau^- \rightarrow \pi^- \pi^- \pi^+ \nu_\tau$  decay mode and found to be in good agreement with the experimental data [16], are not sensitive to the  $A_1$  mass and width parameters since they cancel in the ratios. On the other hand, a measurement of the  $Q^2$  dependence of the structure functions itself would be very sensitive to the  $A_1$  parameters [17] (after having fixed the  $\rho$  parameters in the ratios). An  $A_1$  width of 0.475 GeV predicts a branching fraction  $\mathcal{B}(\tau^- \rightarrow \pi^- \pi^- \pi^+ \nu_\tau)$  of 8.6 % (compared to 6.3 % for  $\Gamma_{A_1} = 599 \text{ GeV}$ ), which seems to be favored by the experimental results for  $\mathcal{B}(3h\nu_\tau) = \mathcal{B}(\pi^- \pi^- \pi^+ \nu_\tau) + \mathcal{B}(K^- \pi^- \pi^+ \nu_\tau) + \mathcal{B}(K^- \pi^- K^+ \nu_\tau) + \mathcal{B}(K^- K^- K^+ \nu_\tau) = 9.51 \pm 0.18$  [12].

There are two relevant particles which have the right quantum numbers in the case of the three particle resonances with strangeness, *i.e.* the  $K_1(1270)$  and the  $K_1(1400)$ . Results from the TPC/Two-Gamma Collaboration had indicated that only the  $K_1(1400)$  occurs in  $\tau$  decays into  $K\pi\pi$  [18]. Thus in [3] only the  $K_1(1400)$  was included. Recent experimental data, however, do not support the results from [18], but rather indicate that both  $K_1$  resonances are produced, with substantially more  $K_1(1270)$  than  $K_1(1400)$  [19]. In fact, this is much more natural within the framework of the resonance enhanced chiral calculation. There are two types of decay chains which lead to the  $K\pi\pi$  states, *i.e.*

- a)  $K_1 \rightarrow K^* \pi, K^* \rightarrow K \pi$ ,
- b)  $K_1 \rightarrow \rho K, \rho \rightarrow \pi \pi$ .

The  $K_1(1400)$  decays exclusively into  $K^* \pi$ , while the  $K_1(1270)$  decays mainly into  $K\rho$ , with a smaller branching ratio into  $K^* \pi$ . Thus one should assume that the decay chain (a) is dominated by the  $K_1(1400)$  with a little admixture of  $K_1(1270)$ , whereas in (b) only the  $K_1(1270)$  contributes:

$$\begin{aligned} T_{K_1}^{(a)}(s) &= \frac{1}{1 + \xi} [\text{BW}_{K_1(1400)}(s) + \xi \text{BW}_{K_1(1270)}(s)] , \\ T_{K_1}^{(b)}(s) &= \text{BW}_{K_1(1270)}(s) . \end{aligned} \quad (32)$$

Using the ratio  $\Gamma(K_1(1270) \rightarrow K^* \pi) / \Gamma(K_1(1400) \rightarrow K^* \pi)$ , we obtain after phase space correction:

$$|\xi| = 0.33. \quad (33)$$

Thus  $\xi$  is determined up to a sign ambiguity. We will consider both possibilities below and find that  $\xi = -0.33$  leads to decay rates which strongly disagree with experimental data (see Sec. VII). Therefore we believe that  $\xi = +0.33$  is the physical choice. In case of the  $K_1$  resonances we use normalized Breit-Wigner propagators with constant widths:

$$BW_{K_1}[s] \equiv \frac{-m_{K_1}^2 + im_{K_1}\Gamma_{K_1}}{[s - m_{K_1}^2 + im_{K_1}\Gamma_{K_1}]}, \quad (34)$$

and [13]

$$\begin{aligned} m_{K_1}(1400) &= 1.402 \text{ GeV} , \quad \Gamma_{K_1}(1400) = 0.174 \text{ GeV} , \\ m_{K_1}(1270) &= 1.270 \text{ GeV} , \quad \Gamma_{K_1}(1270) = 0.090 \text{ GeV} . \end{aligned} \quad (35)$$

All matrix elements of the axial weak current are summarized in Tab. I. The parameterization for the decay modes with two neutral kaons will be discussed in Sec. VI.

## V. ISOSPIN AND FLAVOUR SYMMETRY RELATIONS IN THE VECTOR CURRENT

We consider now the vector current with its contributions from the anomaly. Let us first correct an error in Tab. II of [3]. The two kaon resonance in the decay channels  $K^-\pi^-K^+$  and  $K^0\pi^-\bar{K}^0$  can not be a  $\rho$  resonance. Due to  $G$  parity conservation, the vector resonance  $V$  in the decay chain  $W^- \rightarrow \rho^-$ ,  $\rho^- \rightarrow V\pi^-$ ,  $V \rightarrow (K^-K^+, K^0\bar{K}^0)$  must have  $G = -1$ . Thus only the  $\omega$  and the  $\Phi$  qualify. In the limit of exact  $SU(3)$  flavour symmetry, the coupling  $\rho\Phi\pi$  vanishes, such that only the  $\omega$  contributes in this limit. Taking  $\omega\Phi$  mixing into account, we use

$$T_\omega(s) = \frac{1}{1+\epsilon}[\text{BW}_\omega(s) + \epsilon\text{BW}_\Phi(s)] , \quad (36)$$

where [20]

$$\epsilon = 0.05 . \quad (37)$$

Obviously, use of the  $\omega$  instead of the  $\rho$  changes the relevant isospin symmetry relations.

We use Breit-Wigner propagators with fixed widths, of the same form as the one used for the  $K_1$  resonances in Eq. (34), with [13]

$$\begin{aligned} m_\omega &= 0.782 \text{ GeV} , \quad \Gamma_\omega = 0.00843 \text{ GeV} , \\ m_\phi &= 1.020 \text{ GeV} , \quad \Gamma_\phi = 0.00443 \text{ GeV} . \end{aligned} \quad (38)$$

Similarly to the case of the axial-vector current, there are also different decay chains in the vector current case which have either  $K^*$  or  $\rho/\omega$  two particle resonances. Whereas in the case of the axial-vector current, the relative strength of the strangeness  $S = 0$  and  $S = -1$  resonances could be fixed by the chiral limit in [3], an additional free parameter  $\alpha$  was introduced in the case of the vector current, parameterizing the relative strength of the two body resonances. In the case of the  $K^-\pi^-K^+$  and  $K^0\pi^-\bar{K}^0$  states, the parameterization in [3] was

$$G_3^{(K^-\pi^-K^+)} = G_3^{(K^0\pi^-\bar{K}^0)} = T_\rho^{(2)}(Q^2) \frac{T_\rho^{(1)}(s_2) + \alpha T_{K^*}(S_1)}{1+\alpha} . \quad (39)$$

where  $T_\rho^{(1)}$  should read  $T_\omega$ , as we have explained above. However, by writing the relevant vertices in a flavour invariant way,

$$g_{VV\pi}\epsilon^{\mu\nu\alpha\beta}\text{tr}\left(\partial_\mu V_\nu\partial_\alpha V_\beta\Pi\right)$$

$$ig_{V\pi\pi}\text{tr}\left(V_\mu[\partial^\mu\Pi,\Pi]\right) \quad (40)$$

(where  $\Pi = \pi^a\lambda^a$  describes the pseudoscalar mesons, and similarly  $V_\mu$  parameterizes the vector mesons) the parameter  $\alpha$  can be determined, with the result

$$\alpha = 1/\sqrt{2}. \quad (41)$$

Furthermore, as in the case of the axial-vector current, there are isospin symmetry relations between the matrix elements.

Our final result for the matrix elements of the vector current, taking into account all isospin and flavour symmetry relations, can be found in Tab. II. Note that there are two cases in which the anomaly contribution vanishes in the low energy limit, viz.  $K^-\pi^0K^0$  (because “anomalies do not develop second class currents”, see also [9]) and  $\pi^0\pi^0K^-$  (because of Bose symmetry and the antisymmetry of the anomaly). In both cases, however, the presence of resonances leads to non-vanishing contributions in the higher energy regime. Their precise forms are predicted from other matrix elements which are non zero in the chiral limit and to which they are related by isospin symmetry. These contributions were not taken into account in the parameterization used in [3].

Let us now discuss the three particle vector resonances  $T_\rho^{(2)}$  and  $T_{K^*}^{(2)}$ . In [3], a form for  $T_\rho^{(2)}$  including  $\rho$ ,  $\rho'$  and  $\rho''$  was used, which was obtained from a fit to  $e^+e^- \rightarrow \eta\pi\pi$  data [21,10]. In the three particle vector resonance with strangeness, only the  $K^*(892)$  was included. However, the higher radials will also be included in this paper and we will therefore use

$$T_\rho^{(2)} = \frac{1}{1+\lambda+\mu} [\text{BW}_\rho(s) + \lambda\text{BW}_{\rho'}(s) + \mu\text{BW}_{\rho''}(s)],$$

$$T_{K^*}^{(2)} = \frac{1}{1+\lambda+\mu} [\text{BW}_{K^*}(s) + \lambda\text{BW}_{K^{*\prime}}(s) + \mu\text{BW}_{K^{*\prime\prime}}(s)], \quad (42)$$

where

$$\lambda = \frac{6.5}{-26} = -0.25, \quad \mu = \frac{1}{-26} = -0.038. \quad (43)$$

For the  $\rho$ 's,

$$m_\rho = 0.773 \text{ GeV}, \Gamma_\rho = 0.145 \text{ GeV},$$

$$m_{\rho'} = 1.500 \text{ GeV}, \Gamma_{\rho'} = 0.220 \text{ GeV},$$

$$m_{\rho''} = 1.750 \text{ GeV}, \Gamma_{\rho''} = 0.120 \text{ GeV}. \quad (44)$$

This is exactly the parameterization of [10], which was used in [3], written in a slightly different way. For the  $K^*$ 's, we use the values [13]:

$$m_{K^*} = 0.892 \text{ GeV}, \Gamma_{K^*} = 0.050 \text{ GeV},$$

$$m_{K^{*\prime}} = 1.412 \text{ GeV}, \Gamma_{K^{*\prime}} = 0.227 \text{ GeV},$$

$$m_{K^{*\prime\prime}} = 1.714 \text{ GeV}, \Gamma_{K^{*\prime\prime}} = 0.323 \text{ GeV}. \quad (45)$$

We use energy dependent widths here.

Similar remarks as have been made above for the  $T_{K^*}^{(1)}$  apply here as well. The parameters  $\lambda$  and  $\mu$ , as well as the  $\rho'$  and  $\rho''$  parameters of  $T_{\rho}^{(2)}$  have been obtained from a fit to data. For the  $K^*$  in the anomalous channel, we use the same parameters  $\lambda$  and  $\mu$  as for the  $\rho$  with  $K^{*'}$ ,  $K^{*''}$  parameters taken from [13]. In principle, a more reliable determination of  $T_{K^*}^{(2)}$  could be obtained from a fit to suitable data. It should be noted, however, that the numerical significance of these details is fairly small, because of the small vector channel contribution to the relevant decay modes (see the last 3 entries in column 3 of Tab. III).

## VI. FINAL STATES WITH TWO NEUTRAL KAONS

In [3] the hadronic matrix elements have been expressed in terms of the strong interaction eigenstates  $K^0$  and  $\bar{K}^0$ . The actual measurements, however, are performed in terms of the weak interaction eigenstates  $K_S$  and  $K_L$ . Neglecting  $CP$  violation, the relation between these are given by

$$\begin{aligned} K_S &= \frac{K^0 - \bar{K}^0}{\sqrt{2}} & K_L &= \frac{K^0 + \bar{K}^0}{\sqrt{2}} \\ K_0 &= \frac{K_L + K_S}{\sqrt{2}} & \bar{K}_0 &= \frac{K_L - K_S}{\sqrt{2}} \end{aligned} \quad (46)$$

Since the weak current  $J^\mu$  produces states with  $|S| = 0, 1$  only, *i.e.*

$$\langle K^0 \pi^- K^0 | J^\mu | 0 \rangle = \langle \bar{K}^0 \pi^- \bar{K}^0 | J^\mu | 0 \rangle = 0 \quad (47)$$

we can easily show that

$$\begin{aligned} &\langle K_L(q_1) \pi^-(q_2) K_L(q_3) | J^\mu | 0 \rangle = - \langle K_S(q_1) \pi^-(q_2) K_S(q_3) | J^\mu | 0 \rangle \\ &= \frac{1}{2} \left\{ \langle K^0(q_1) \pi^-(q_2) \bar{K}^0(q_3) | J^\mu | 0 \rangle + \langle \bar{K}^0(q_1) \pi^-(q_2) K^0(q_3) | J^\mu | 0 \rangle \right\} , \\ &\langle K_S(q_1) \pi^-(q_2) K_L(q_3) | J^\mu | 0 \rangle \\ &= \frac{1}{2} \left\{ \langle K^0(q_1) \pi^-(q_2) \bar{K}^0(q_3) | J^\mu | 0 \rangle - \langle \bar{K}^0(q_1) \pi^-(q_2) K^0(q_3) | J^\mu | 0 \rangle \right\} . \end{aligned} \quad (48)$$

Let us start with the discussion of the axial part of the weak current. From Eq. (20) and Tab. I, we have

$$\begin{aligned} &\langle K^0 \pi^- \bar{K}^0 | A^\mu | 0 \rangle = \frac{2\sqrt{2}}{3f_\pi} \left( \frac{-\cos \theta_c}{2} \right) \text{BW}_{A_1}(Q^2) \\ &\times \left\{ T_{\rho}^{(1)}(s_2)(q_1 - q_3)_\nu + T_{K^*}(s_1)(q_2 - q_3)_\nu \right\} T^{\mu\nu} . \end{aligned} \quad (49)$$

Using Eq. (48) one obtains

$$\begin{aligned}
& \langle K_L(q_1)\pi^-(q_2)K_L(q_3)|A^\mu|0 \rangle = - \langle K_S(q_1)\pi^-(q_2)K_S(q_3)|A^\mu|0 \rangle = \\
& \frac{2\sqrt{2}\cos\theta_C}{3f_\pi} \text{BW}_{A_1}(Q^2) \left\{ T_{K^*}^{(1)}(s_3)(q_1 - q_3)_\nu - [T_{K^*}^{(1)}(s_1) + T_{K^*}^{(1)}(s_3)](q_2 - q_3)_\nu \right\} T^{\mu\nu}, \\
& \langle K_S(q_1)\pi^-(q_2)K_L(q_3)|A^\mu|0 \rangle = \frac{2\sqrt{2}}{3f_\pi} \left( \frac{-\cos\theta_C}{4} \right) \text{BW}_{A_1}(Q^2) \\
& \times \left\{ [2T_\rho^{(1)}(s_2) + T_{K^*}^{(1)}(s_3)](q_1 - q_3)_\nu + [T_{K^*}^{(1)}(s_1) - T_{K^*}^{(1)}(s_3)](q_2 - q_3)_\nu \right\} T^{\mu\nu}. \quad (50)
\end{aligned}$$

We turn now to the matrix element of the vector current. According to Tab. I, the relevant matrix element is

$$\langle K^0\pi^-\overline{K^0}|V^\mu|0 \rangle = \frac{i}{2\sqrt{2}\pi^2 f_\pi^3} \cos\theta_c T_\rho^{(2)}(Q^2)(\sqrt{2}-1)[\sqrt{2}T_\omega(s_2) + T_{K^*}^{(1)}(s_1)]\epsilon^{\mu\alpha\beta\gamma}q_{1\alpha}q_{2\beta}q_{3\gamma}. \quad (51)$$

In this case Eq. (48) yields (see Tab. II)

$$\begin{aligned}
& \langle K_L\pi^-K_L|V^\mu|0 \rangle = - \langle K_S\pi^-K_S|V^\mu|0 \rangle \\
& = \frac{i}{2\sqrt{2}\pi^2 f_\pi^3} \frac{\cos\theta_c}{2} (\sqrt{2}-1) T_\rho^{(2)}(Q^2) [T_{K^*}(s_1) - T_{K^*}(s_3)] \epsilon^{\mu\alpha\beta\gamma} q_{1\alpha} q_{2\beta} q_{3\gamma}, \\
& \langle K_S\pi^-K_L|V^\mu|0 \rangle \\
& = \frac{i}{2\sqrt{2}\pi^2 f_\pi^3} \frac{\cos\theta_c}{2} (\sqrt{2}-1) T_\rho^{(2)}(Q^2) [2\sqrt{2}T_\omega(s_2) + T_{K^*}^{(1)}(s_1) + T_{K^*}^{(1)}(s_3)] \epsilon^{\mu\alpha\beta\gamma} q_{1\alpha} q_{2\beta} q_{3\gamma}. \quad (52)
\end{aligned}$$

Whereas the relative amounts of  $K_S K_S$  and  $K_L K_L$  states are fixed by general symmetry considerations to be equal

$$\frac{\Gamma(K_S\pi^-K_S)}{\Gamma(K_L\pi^-K_L)} = 1, \quad (53)$$

the ratio  $R$  of  $K_S K_S$  and  $K_S K_L$  states

$$R = \frac{\Gamma(K_S\pi^-K_S)}{\Gamma(K_S\pi^-K_L)}, \quad (54)$$

is model dependent and will be discussed in the next section.

## VII. NUMERICAL RESULTS

After having fixed our model for the form factors, we next present numerical results for the hadronic decay widths  $\Gamma(abc)$  normalized to the leptonic width  $\Gamma_e$  and for the branching ratios in Tab. III. To calculate the branching ratios, we use the theoretical prediction for  $\Gamma_e/\Gamma_{\text{tot}} = 17.8\%$  based on the experimental values for the tau mass  $m_\tau = 1.7771 \text{ GeV}$  and lifetime  $\tau_\tau = 291.6 \text{ fs}$  [22], rather than using the experimental branching ratio.

The decay rate for  $\tau$  decays into three mesons can be calculated from

$$\Gamma(\tau \rightarrow 3h) = \frac{G^2}{12m_\tau} \left( \frac{\cos \theta_c}{\sin \theta_c} \right)^2 \frac{1}{(4\pi)^5} \int \frac{dQ^2}{Q^4} ds_1 ds_2 (m_\tau^2 - Q^2)^2 \left( 1 + \frac{2Q^2}{m_\tau^2} \right) (W_A + W_B) \quad (55)$$

where

$$W_A = (x_1^2 + x_3^2) |F_1|^2 + (x_2^2 + x_3^2) |F_2|^2 + 2(x_1 x_2 - x_3^2) \text{Re}(F_1 F_2^*) \quad (56)$$

$$W_B = x_4^2 |F_3|^2 \quad (57)$$

The variables  $x_i$  are defined by  $x_1 = V_1^x = q_1^x - q_3^x$ ,  $x_2 = V_2^x = q_2^x - q_3^x$ ,  $x_3 = V_1^y = q_1^y = -q_2^y$ ,  $x_4 = V_3^z = \sqrt{Q^2} x_3 q_3^x$ , where  $q_i^x$  ( $q_i^y$ ) denotes the  $x$  ( $y$ ) component of the momentum of meson  $i$  in the hadronic rest frame as introduced in [5,6]. They can easily be expressed in terms of  $s_1$ ,  $s_2$  and  $s_3$  [5]. Eq. (55) shows that there is no interference between the axial vector contributions ( $F_1, F_2$ ) and the vector current contribution ( $F_3$ ) in the total decay width.

Our numerical results for the normalized decay widths  $\Gamma(abc)/\Gamma_e$  and the branching ratios  $\mathcal{B}(abc)$  for our preferred parameter choices are given in Tab. III for the various decay channels  $abc$ . To get a feeling for the numerical importance of the vector current (*i.e.* the “anomaly”), we list its contribution to the decay width in column 3 of Tab. III. For comparison, we have also listed the available experimental data in column 5 of Tab. III.

Our numerical results for  $\mathcal{B}(K_S \pi^- K_S)$ ,  $\mathcal{B}(K^- \pi^- \pi^+)$  and  $\mathcal{B}(\pi^- \bar{K}^0 \pi^0)$  appear to be considerably higher than the experimental results, whereas the other predictions agree fairly well.

Our results for the  $K^- \pi^- K^+$  final state do not agree with the results in [10], where the contribution of the axial-vector channel amounts to less than 10% to the decay rate in this channel. In fact, our predictions for the axial-vector contribution is about 60%. This result is fairly insensitive towards the details of the  $K^*$  parameterization (see Tab. III, V and VI, as discussed below). It is however sensitive towards the  $A_1$  parameters. Use of  $\Gamma_{A_1} = 0.599$  GeV in Eq. (31) reduces the axial-vector contribution to about 48%, which is still considerably larger than the value in [10].

Let us now comment on the sign of parameter  $\xi$  in the parameterization of the strange axial resonances  $K_1$  in Eq. (33), which is only determined up to a sign ambiguity. Our predictions for the Cabibbo suppressed decays in Tab. III are obtained for  $\xi = +0.33$ . For comparison, Tab. IV shows the results for  $\xi = -0.33$ . The choice  $\xi = +0.33$  is clearly preferred by the experimental data and we therefore believe that this is the physical choice.

Next we discuss our choice of the two body vector resonance with strangeness, *i.e.* the parameterization of  $T_{K^*}^{(1)}$  in Eq. (10). We have used a  $K^{*'}(1410)$  contribution in  $T_{K^*}^{(1)}$  with a strength of  $\beta_{K^*} = -0.135$  relative to the  $K^*(892)$ , as determined from  $\tau \rightarrow K^* \nu_\tau$  (see Sec. II). Use of  $\beta_{K^*} = -0.11$ , which is also consistent with the  $\tau \rightarrow K^* \nu_\tau$  decay rate leads to the results in Tab. V. The results are fairly close to the numbers presented in Tab. III. Use of  $\beta_{K^*} = 0$  as in [3] leads to the results in Tab. VI, which overall agree better with the experimental results than those obtained with  $\beta_{K^*} = -0.135$ .

At this point a few comments are in order. Firstly, by fixing the coupling constants from the chiral limit, we effectively assume exact  $SU(3)$  flavour symmetry for the coupling

constants. The most important effects of flavour symmetry breaking have been taken into account by using the physical masses and decay widths of the mesons in the propagators and in the phase space. However, one could expect additional explicit flavour symmetry breaking in the couplings. The approximation made may lead to an error of about 10–30% on the matrix element level. Thus we consider the agreement between theory and experiment in Tab. III as reasonable. Note that this approximation will mainly lead to an overall normalization error of the form factors rather than to a modification of their momentum dependence in the relevant physical region.

Secondly, we remind the reader that a change of  $\beta_{K^*}$  has two different effects. It changes the normalization of the  $K^*(892)$  contribution, which is proportional to  $1/(1 + \beta_{K^*})^2$  in the rate, and it changes the size of the  $K^*(1410)$  contribution, which is however strongly phase space suppressed. The main effect of choosing a different  $\beta_{K^*}$  is to change the normalization, which may well compensate the error being made by deriving the coupling constants using flavour symmetry.

Thirdly, we wish to emphasize that in spite of this normalization uncertainty of the matrix elements, our parameterizations are by no means arbitrary. We believe that they give a very good description of the resonance substructures which determine the decay rates as well as differential distributions and structure functions.

Our choice for  $\Gamma_{A_1} = 0.475$  GeV was already discussed before [see the discussion after Eq. (31)]. An  $A_1$  width of  $\Gamma_{A_1} = 0.599$  GeV decreases the branching fractions for the  $K\pi K$  decay modes in Tab. III by about 15%, which is of course entirely due to a decrease of the axial-vector contribution (see Tab. I).

As already mentioned in Sec. VI, the decay rates for  $K_S\pi^-K_S$  and  $K_L\pi^-K_L$  are the same. This follows immediately from Eqs. (46,47) and is a model independent statement. On the other hand, the ratio  $R$

$$R = \frac{\Gamma(K_S\pi^-K_S)}{\Gamma(K_S\pi^-K_L)}, \quad (58)$$

is model dependent and we obtain from Tab. III

$$R = 0.48 \quad (59)$$

This number is fairly insensitive towards a variation of the  $K^*$  resonance parameters. However, use of  $\Gamma_{A_1} = .599$  GeV in Eq. (31) would lead to  $R = 0.42$ .

We would like to point out that a study of angular correlations of the hadronic system allows for much more detailed studies of the hadronic charged current (including the details of the two and three body resonance parameters) than it is possible by rate measurements alone. Of particular interest is the angular distribution of the three mesons in the three meson rest frame. The distribution of the normal on the hadronic plane with respect to  $\vec{n}_L$  (the direction of the laboratory as seen from the hadronic rest frame) allows for a model independent separation of the axial-vector and the vector current contribution, *i.e.* the structure functions  $W_A(Q^2, s_1, s_2, s_3)$  and  $W_B(Q^2, s_1, s_2, s_3)$  in Eq. (55) can be determined separately even without reconstructing the  $\tau$  rest frame [5]. An experimental analysis of the  $Q^2$  and  $s_i$  distributions of these structure functions would clearly help to test the parameterizations in Tab. I and Tab. II unambiguously. More general distributions like the rotation of the mesons around the normal, allow for even more detailed studies of the hadronic matrix

element [5].  $Q^2$  distribution for structure functions for  $K^-\pi^-K^+$ ,  $K^-\pi^-\pi^+$  and  $\eta\pi^-\pi^+$  final states have already been presented in [23] based on the parameterization in [3]. We will study the  $Q^2$  and the full Dalitz plot distributions for the decay rates and the two structure functions  $W_A$  and  $W_B$  based on the model in this paper in a future publication.

### VIII. SUMMARY AND CONCLUSIONS

We have discussed tau decays into final states with one or two kaons. The decays into  $\pi K$  are dominated by the  $K^*$  resonance and therefore allow for a determination of the parameters of the  $K^*$  propagator. The experimental branching ratio can be used to obtain a rough estimate of the  $K^*(1410)$  contribution, but we would like to urge for a detailed study of the invariant mass distribution of the hadronic system in order to measure the strength of its contribution. The decay into two kaons is predicted in agreement with experimental data assuming dominance by the high mass tail of the  $\rho$ .

The three meson final states  $K\pi K$  and  $\pi\pi K$  allow for a much more involved resonance substructure. We have extensively reanalyzed these on the basis of the model of [3]. Our final results for the branching ratios with our preferred parameter choices have been given in Tab. III. They compare reasonably with experimental data, but are rather sensitive to parameters such as  $\beta_{K^*}$ .

In the case of two neutral kaons, we have expressed the matrix elements in terms of the  $K_S$  and  $K_L$  states and given a prediction for the model dependent ratio of the  $K_S\pi^-K_S$  and  $K_S\pi^-K_L$  final states.

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A fortran code for the three meson matrix elements is available from the authors [mirkes@phenom.physics.wisc.edu]. The code allows for a straightforward implementation in the TAUOLA Monte Carlo program.



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# TABLES

TABLE I. Parameterization of the form factors  $F_1$  and  $F_2$  in Eqs. (23,24) for the matrix elements of the weak axial-vector current for the various channels.

channel (abc)	$A^{(abc)}$	$G_1^{(abc)}(Q^2, s_2, s_3)$	$G_2^{(abc)}(Q^2, s_1, s_3)$
$K^- \pi^- K^+$	$\frac{-\cos \theta_c}{2}$	$\text{BW}_{A_1}(Q^2) T_\rho^{(1)}(s_2)$	$\text{BW}_{A_1}(Q^2) T_{K^*}^{(1)}(s_1)$
$K^0 \pi^- \bar{K}^0$	$\frac{-\cos \theta_c}{2}$	$\text{BW}_{A_1}(Q^2) T_\rho^{(1)}(s_2)$	$\text{BW}_{A_1}(Q^2) T_{K^*}^{(1)}(s_1)$
$K_S \pi^- K_S$	$\frac{-\cos \theta_c}{4}$	$\text{BW}_{A_1}(Q^2) T_{K^*}^{(1)}(s_3)$	$-\text{BW}_{A_1}(Q^2) [T_{K^*}^{(1)}(s_1) + T_{K^*}^{(1)}(s_3)]$
$K_S \pi^- K_L$	$\frac{-\cos \theta_c}{4}$	$\text{BW}_{A_1}(Q^2) [2T_\rho^{(1)}(s_2) + T_{K^*}^{(1)}(s_3)]$	$\text{BW}_{A_1}(Q^2) [T_{K^*}^{(1)}(s_1) - T_{K^*}^{(1)}(s_3)]$
$K^- \pi^0 K^0$	$\frac{3 \cos \theta_c}{2\sqrt{2}}$	$\text{BW}_{A_1}(Q^2) \left[ \frac{2}{3} T_\rho^{(1)}(s_2) + \frac{1}{3} T_{K^*}^{(1)}(s_3) \right]$	$\frac{1}{3} \text{BW}_{A_1}(Q^2) [T_{K^*}^{(1)}(s_1) - T_{K^*}^{(1)}(s_3)]$
$\pi^0 \pi^0 K^-$	$\frac{\sin \theta_c}{4}$	$T_{K_1}^{(a)}(Q^2) T_{K^*}^{(1)}(s_2)$	$T_{K_1}^{(a)}(Q^2) T_{K^*}^{(1)}(s_1)$
$K^- \pi^- \pi^+$	$\frac{-\sin \theta_c}{2}$	$T_{K_1}^{(a)}(Q^2) T_{K^*}^{(1)}(s_2)$	$T_{K_1}^{(b)}(Q^2) T_\rho^{(1)}(s_1)$
$\pi^- \bar{K}^0 \pi^0$	$\frac{3 \sin \theta_c}{2\sqrt{2}}$	$\frac{2}{3} T_{K_1}^{(b)}(Q^2) T_\rho^{(1)}(s_2) + \frac{1}{3} T_{K_1}^{(a)}(Q^2) T_{K^*}^{(1)}(s_3)$	$\frac{1}{3} T_{K_1}^{(a)}(Q^2) [T_{K^*}^{(1)}(s_1) - T_{K^*}^{(1)}(s_3)]$

TABLE II. Parameterization of the form factor  $F_3$  in Eq. (25) for the matrix elements of the weak vector current for the various channels.

channel (abc)	$A^{(abc)}$	$G_3^{(abc)}(Q^2, s_1, s_2, s_3)$
$K^- \pi^- K^+$	$-\cos \theta_c$	$T_\rho^{(2)}(Q^2)(\sqrt{2}-1) [\sqrt{2}T_\omega(s_2) + T_{K^*}^{(1)}(s_1)]$
$K^0 \pi^- \overline{K}^0$	$\cos \theta_c$	$T_\rho^{(2)}(Q^2)(\sqrt{2}-1) [\sqrt{2}T_\omega(s_2) + T_{K^*}^{(1)}(s_1)]$
$K_S \pi^- K_S$	$\frac{-\cos \theta_c}{2}$	$T_\rho^{(2)}(Q^2)(\sqrt{2}-1) [T_{K^*}^{(1)}(s_1) - T_{K^*}^{(1)}(s_3)]$
$K_S \pi^- K_L$	$\frac{\cos \theta_c}{2}$	$T_\rho^{(2)}(Q^2)(\sqrt{2}-1) [2\sqrt{2}T_\omega(s_2) + T_{K^*}^{(1)}(s_1) + T_{K^*}^{(1)}(s_3)]$
$K^- \pi^0 K^0$	$\frac{-\cos \theta_c}{\sqrt{2}}$	$T_\rho^{(2)}(Q^2)(\sqrt{2}-1) [T_{K^*}^{(1)}(s_3) - T_{K^*}^{(1)}(s_1)]$
$\pi^0 \pi^0 K^-$	$\sin \theta_c$	$\frac{1}{4}T_{K^*}^{(2)}(Q^2) [T_{K^*}^{(1)}(s_1) - T_{K^*}^{(1)}(s_2)]$
$K^- \pi^- \pi^+$	$\sin \theta_c$	$\frac{1}{2}T_{K^*}^{(2)}(Q^2) [T_\rho^{(1)}(s_1) + T_{K^*}^{(1)}(s_2)]$
$\pi^- \overline{K}^0 \pi^0$	$\sqrt{2} \sin \theta_c$	$\frac{1}{4}T_{K^*}^{(2)}(Q^2) [2T_\rho^{(1)}(s_2) + T_{K^*}^{(1)}(s_1) + T_{K^*}^{(1)}(s_3)]$

TABLE III. Predictions for the normalized decay widths  $\Gamma(abc)/\Gamma_e$  and the branching ratios  $\mathcal{B}(abc)$  for the various channels with  $\beta_{K^*} = -0.135$  in Eq. (10). The contribution from the vector current is listed in column 3 and available experimental data are listed in column 5. The later are taken from [12], except for  $K_S\pi^-K_S$ , where we quote the value for  $K_S h K_S$  given in [19].

channel (abc)	$\left(\frac{\Gamma(abc)}{\Gamma_e}\right)^{(pred.)}$	$\left(\frac{\Gamma(abc)}{\Gamma_e}\right)^{(pred.)}_{anomaly}$	$(\mathcal{B}(abc))^{(pred.)} [\%]$	$(\mathcal{B}(abc))^{(expt.)} [\%]$
$K^-\pi^-K^+$	0.011	0.0045	0.20	$0.20 \pm 0.07$
$K^0\pi^-\overline{K}^0$	0.011	0.0045	0.20	
$K_S\pi^-K_S$	0.0027	0.0008	0.048	$0.021 \pm 0.006$
$K_S\pi^-K_L$	0.0058	0.0029	0.10	
$K^-\pi^0K^0$	0.0090	0.0032	0.16	$0.12 \pm 0.04$
$\pi^0\pi^0K^-$	0.0080	0.0007	0.14	$0.09 \pm 0.03$
$K^-\pi^-\pi^+$	0.043	0.0043	0.77	$0.40 \pm 0.09$
$\pi^-\overline{K}^0\pi^0$	0.054	0.0058	0.96	$0.41 \pm 0.07$

TABLE IV. Same as Tab. III for the Cabibbo suppressed decays with  $\xi = -0.33$  in Eq. (33).

channel (abc)	$\left(\frac{\Gamma(abc)}{\Gamma_e}\right)^{(pred.)}$	$\left(\frac{\Gamma(abc)}{\Gamma_e}\right)^{(pred.)}_{anomaly}$	$(\mathcal{B}(abc))^{(pred.)} [\%]$	$(\mathcal{B}(abc))^{(expt.)} [\%]$
$\pi^0\pi^0K^-$	0.017	0.0007	0.30	$0.09 \pm 0.03$
$K^-\pi^-\pi^+$	0.077	0.0043	1.37	$0.40 \pm 0.09$
$\pi^-\overline{K}^0\pi^0$	0.087	0.0058	1.55	$0.41 \pm 0.07$

TABLE V. Same as Tab. III but with  $\beta_{K^*} = -0.11$  in Eq. (10).

channel (abc)	$\left(\frac{\Gamma(abc)}{\Gamma_e}\right)^{(pred.)}$	$\left(\frac{\Gamma(abc)}{\Gamma_e}\right)^{(pred.)}_{anomaly}$	$(\mathcal{B}(abc))^{(pred.)} [\%]$	$(\mathcal{B}(abc))^{(expt.)} [\%]$
$K^-\pi^-K^+$	0.0106	0.0042	0.19	$0.20 \pm 0.07$
$K^0\pi^-\overline{K}^0$	0.0106	0.0042	0.19	
$K_S\pi^-K_S$	0.0026	0.0007	0.046	$0.021 \pm 0.006$
$K_S\pi^-K_L$	0.0055	0.0027	0.098	
$K^-\pi^0K^0$	0.0085	0.0030	0.15	$0.12 \pm 0.04$
$\pi^0\pi^0K^-$	0.0076	0.0007	0.14	$0.09 \pm 0.03$
$K^-\pi^-\pi^+$	0.041	0.0041	0.74	$0.40 \pm 0.09$
$\pi^-\overline{K}^0\pi^0$	0.052	0.0056	0.93	$0.41 \pm 0.07$

TABLE VI. Same as Tab. III but with  $\beta_{K^*} = 0$  in Eq. (10).

channel (abc)	$\left(\frac{\Gamma(abc)}{\Gamma_e}\right)^{(pred.)}$	$\left(\frac{\Gamma(abc)}{\Gamma_e}\right)^{(pred.)}_{anomaly}$	$(\mathcal{B}(abc))^{(pred.)} [\%]$	$(\mathcal{B}(abc))^{(expt.)} [\%]$
$K^-\pi^-K^+$	0.0084	0.0034	0.15	$0.20 \pm 0.07$
$K^0\pi^-\overline{K}^0$	0.0084	0.0034	0.15	
$K_S\pi^-K_S$	0.0020	0.0006	0.036	$0.021 \pm 0.006$
$K_S\pi^-K_L$	0.0044	0.0022	0.078	
$K^-\pi^0K^0$	0.0067	0.0024	0.12	$0.12 \pm 0.04$
$\pi^0\pi^0K^-$	0.0060	0.0005	0.11	$0.09 \pm 0.03$
$K^-\pi^-\pi^+$	0.035	0.0035	0.62	$0.40 \pm 0.09$
$\pi^-\overline{K}^0\pi^0$	0.045	0.0048	0.81	$0.41 \pm 0.07$